Multiple integral - I

Double integrals :-

Let $f(x,y)$ be a continuous and single valued function of x and y with in a region R bounded by a closed curve c and upon the boundary c. Let the region R be subdivided in any manner into n-subregions of areas $\Delta A_1, \Delta A_2 \cdots \Delta A_n$.

Let (ξ_r, η_r) be any point in the sub region of area ΔA_r and consider the sum $\sum_{r=1}^{n} f(\xi_r, \eta_r)\Delta A_r$ the limit of this sum as $n \to \infty$ and $\Delta A_r \to 0$ $(r = 1, 2 \cdots\cdots)$ is defined as the $\iint$ double integral of $f(x,y)$ over the region R.

thus $\iint_R f(x,y)\,dA = \lim_{n\to\infty} \sum_{r=1} f(\xi_r, \eta_r)\Delta A_r$.

The region R is called the region of integration corresponding to interval of integration (a,b) in the case of simple integral.

This integral is sometimes written as

$\iint_R f(x,y)\,dx\,dy$.

Evaluation of the double integral :-

The double integral is evaluated by considering $f(x,y)$ as a function of x alone and integrated between but regarding y as a constant and integrating it between $x = f_1(y)$ and $x = f_2(y)$ and then integrating the resulting function of y between $y = c$, $y = d$.

Similarly by taking the sum of the terms in each column and then adding these sums

$$\iint_R f(x,y)\,dA = \int_a^b\left[\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy\right]dx.$$

Problems :

1. Evaluate $\int_0^2\int_0^3 (xy)\,dx\,dy$

$$\int_0^2\int_0^3 xy\,dx\,dy = \int_0^2 x\left[\int_0^3 y\,dy\right]dx$$

$$= \int_0^2 x\left(\left(\frac{y^2}{2}\right)_0^3\right)dx.$$

$$= \int_0^2 x \times \frac{9}{2}\,dx$$

$$= \frac{9}{2}\left[\frac{x^2}{2}\right]_0^2$$

$$= \frac{9}{\cancel{2}} \times \frac{\cancel{4}}{\cancel{2}}$$

$$= 9$$

2. Evaluate $\int_0^2\int_0^x y\,dx\,dy$

$$\int_0^2\int_0^x y\,dx\,dy = \int_0^2\left[\int_0^x dy\right]dx$$

$$= \int_0^2 [y]_0^x\,dx$$

$$= \int_0^2 x\,dx$$

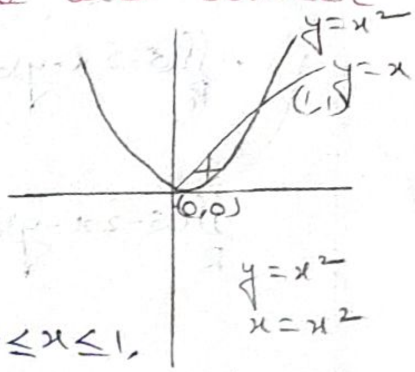
$$= \left[\frac{x^2}{2}\right]_0^2$$

$$= \frac{\cancel{4}\,2}{\cancel{2}}$$

$$= 2$$

3. Evaluate $\int_0^2\int_0^x y\,dx\,dy$

4. Evaluate $\iint xy(x+y)\,dx\,dy$ over the area between $y=x^2$, $y=x$.



The curves $y=x^2$ and $y=x$ intersect at $(0,0)(1,1)$.

The region of integration is $0 \le x \le 1$,

$x^2 \le y \le x$

$y = x^2$

$x = x^2$

$(x - x^2) = 0$

$x(1-x) = 0$

$x = 0,\ x = 1$

$\int_{x=0}^{1}\int_{y=x^2}^{x} (xy)(x+y)\,dx\,dy$

$$\therefore \iint xy(x+y)\,dx\,dy = \int_{x=0}^{1}\int_{y=x^2}^{x} xy(x+y)\,dx\,dy$$

$$= \int_0^1 \left[\int_{x^2}^{x} (x^2y + xy^2)\,dy\right]dx$$

$$= \int_0^1 \left[x^2\frac{y^2}{2} + x\frac{y^3}{3}\right]_{x^2}^{x} dx$$

$$= \int_0^1 \left[x^2\frac{x^2}{2} + x\frac{x^3}{3} - x^2\frac{x^4}{2} - x\frac{x^6}{3}\right]dx$$

$$= \int_0^1 \left[\frac{x^4}{2} + \frac{x^4}{3} - \frac{x^6}{2} - \frac{x^7}{3}\right]dx$$

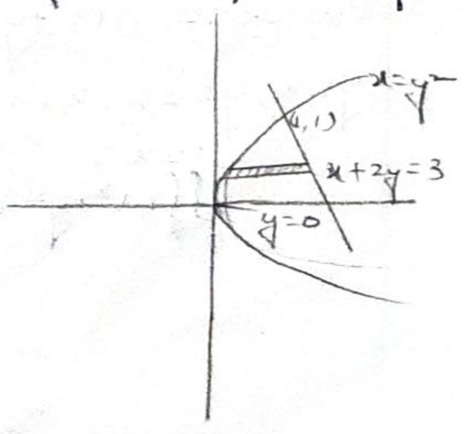
$$= \left[\frac{x^5}{10} + \frac{x^5}{12} - \frac{x^7}{14} - \frac{x^8}{24}\right]_0^1$$

$$= \frac{1}{10} + \frac{1}{12} - \frac{1}{14} - \frac{1}{24}$$

$$= \frac{3}{56}$$

5. Evaluate $\iint_R (5-2x-y)\,dx\,dy$ where R is given by $y=0$, $x+2y=3$, $x=y^2$.

The region R bounded by $y=0$, $x+2y=3$, $x=y^2$ is shown in the fig.



It is convinient to consider horizontal strip, with one end moving on $x=y^2$ and another end moving on $x=3-2y$. To cover the region this strip has to be slided from $y=0, y=1$

$$\iint_R (5-2x-y)\,dx\,dy$$

$y=0, y=1$

$$\iint_R (5-2x-y)\,dx\,dy = \int_0^1 \left[\int_{y^2}^{3-2y} (5-2x-y)\,dx\right] dy$$

$$= \int_0^1 \left[\left(5x - \frac{2x^2}{2} - yx\right)_{y^2}^{3-2y}\right] dy$$

$$= \int_0^1 \Big[5(3-2y) - (3-2y)^2 - y(3-2y) - 5y^2 - y^4 - y^3\Big] dy$$

$$= \int_0^1 (y^4 + y^3 - 7y^2 - y + 6)\,dy$$

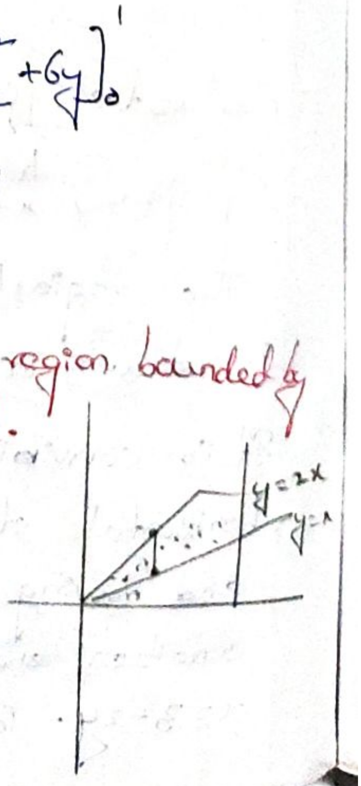
$$= \left[\frac{y^5}{5} + \frac{y^4}{4} - \frac{7y^3}{3} - \frac{y^2}{2} + 6y\right]_0^1$$

$$= \frac{1}{5} + \frac{1}{4} - \frac{7}{3} - \frac{1}{2} + 6$$

$$= \frac{217}{60}$$

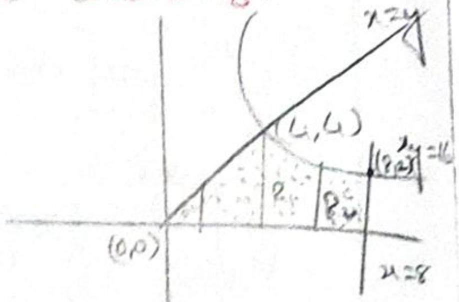
6. Evaluate $\iint (x^2+y^2)\,dx\,dy$ over the region bounded by $y=x, y=2x, x=1$ in the quadrant.

$$\int_0^1 \int_x^{2x} (x^2+y^2)\,dx\,dy$$



P Evaluate $\iint_R x^2\,dx\,dy$ where R is the region in the First quadrant bounded by the hyperbola $xy=16$ and the lines $y=x$, $y=0$ and $x=8$.

The region is shown in the fig as shaded area.



Let us divide the region in two regions R_1, R_2.

Hence

$$\iint_R x^2\,dx\,dy = \iint_{R_1} x^2\,dx\,dy + \iint_{R_2} x^2\,dx\,dy$$

$\therefore$ the limits of y are $(0, \frac{16}{x})$.

this strip must be moved from $x=4$, to $x=8$ to cover R_2

$$\iint_R x^2\,dx\,dy = \iint_{R_1} x^2\,dx\,dy + \iint_{R_2} x^2\,dx\,dy$$

$$= \int_0^4\int_0^x x^2\,dx\,dy + \int_4^8\int_0^{16/x} x^2\,dx\,dy.$$

$$= \int_0^4 x^2[y]_0^x\,dx + \int_4^8 x^2[y]_0^{16/x}\,dx$$

$$= \int_0^4 x^2[x]\,dx + \int_4^8 x^2\left[\frac{16}{x} - 0\right]dx$$

$$= \int_0^4 x^2(x)\,dx + \int_4^8 x^2\left(\frac{16}{x}\right)dx$$

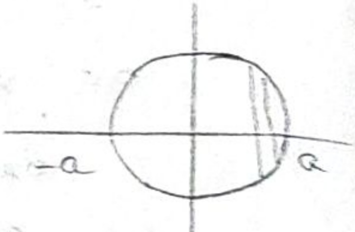
$$= \left[\frac{x^4}{4}\right]_0^4 + \cancel{16}^{8}\left[\frac{x^2}{\cancel{2}}\right]_4^8$$

$$= \frac{256}{4} + 8(64-16)$$

$$= \frac{256}{4} + 384$$

Answer 448

Evaluate $\iint (x+y)^2 \, dx\, dy$ over the area bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.



(3) Given ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}$$

$$y^2=b^2\left[1-\frac{x^2}{a^2}\right]$$

$$=b^2\left[\frac{a^2-x^2}{a^2}\right]$$

$$y=\frac{b^2}{a^2}\left[a^2-x^2\right]$$

$$y=\pm\frac{b}{a}\sqrt{a^2-x^2}$$

the region of integration is $-a\le x\le a$ and

$$-\frac{b}{a}\sqrt{a^2+x^2}\le y\le \frac{b}{a}\sqrt{a^2-x^2}$$

$$\iint_R (x+y)^2\,dx\,dy=\iint_R (x^2+y^2+2xy)\,dx\,dy$$

$$=\int_{-a}^{a}\int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{b/a\sqrt{a^2-x^2}} (x^2+y^2+2xy)\,dx\,dy$$

$$=\int_{-a}^{+a}\left[x^2y+\frac{y^3}{3}+2x\frac{y^2}{2}\right]_{-\frac{b}{a}\sqrt{a^2-x^2}}^{b/a\sqrt{a^2-x^2}} dx$$

$$=\int_{-a}^{a} x^2\left\{\frac{b}{a}\sqrt{a^2-x^2}+\frac{b}{a}\sqrt{a^2+x^2}+\frac{1}{3}\left\{\left(\frac{b}{a}\sqrt{a^2-x^2}\right)^3+\left(\frac{b}{a}\sqrt{a^2-x^2}\right)^3\right\}\right.$$
$$\left.+x\left\{\left(\frac{b}{a}\sqrt{a^2+x^2}\right)^2+\left(\frac{b}{a}\sqrt{a^2+x^2}\right)^2\right\}\right\}$$

$$=\int_{-a}^{a} x^2\left[\frac{2b}{a}\sqrt{a^2-x^2}+\frac{2}{3}\frac{b^3}{a^3}(a^2-x^2)^{3/2}+0\right]dx.$$

$$=2\int_0^a \left(x^2\left(\frac{2b}{a}\sqrt{a^2-x^2}\right)+\frac{2}{3}\frac{b^3}{a^3}(a^2-x^2)^{3/2}\right)dx$$

$$= 4\int_0^{\pi/2} \frac{b}{a} a^2\sin^2\theta\, a\cos\theta + \frac{b^3}{3a^3} a^3\cos^3\theta \Big]\, a\cos\theta\, d\theta$$

put $x = a\sin\theta$

$dx = a\cos\theta\, d\theta$

$$= 4ba^3\int_0^{\pi/2} \sin^2\theta\cos^2\theta\, d\theta + \frac{4ab^3}{3}\int_0^{\pi/2}\cos^4\theta\, d\theta$$

$$= 4a^3b\frac{1}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} + 4ab^3\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$$= (a^3b+ab^3)\frac{\pi}{4}$$

$$= \frac{ab(a^2+b^2)}{4}\pi.$$

P. Evaluate ~~over~~ $\iint xy\,dxdy$ taken over the region positive quadrant of the circle $x^2+y^2=a^2$.

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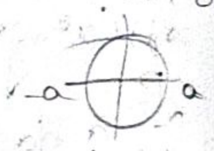
$\int_0^a\int_0^{\frac{b}{a}\sqrt{a^2-x^2}}$ $x^2+y^2=a^2$

$y^2 = a^2-x^2$

$y = \pm\sqrt{a^2-x^2}$

positive area lower lim '0'



$x = (0, a)$

$y = 0, \sqrt{a^2-x^2}$

the region of integration is the positive quadrant of the circle $x^2+y^2=a^2$

$\therefore$ y varies 0 to $\sqrt{a^2-x^2}$ and x varies 0 to a

$$\iint_R xy\,dxdy = \int_0^a\int_0^{\sqrt{a^2-x^2}} xy\,dxdy$$

$$= \int_0^a x\left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a x\,\frac{(a^2-x^2)}{2}\,dx$$

$$= \frac{1}{2}\int_0^a [xa^2 - x^3]\,dx$$

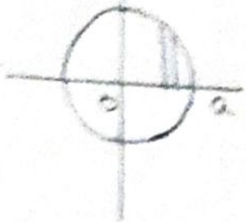
$$= \frac{1}{2}\left[\frac{x^2a^2}{2} - \frac{x^4}{4}\right]_0^a$$

* the region of integration is the positive quadrant of the circle $x^2+y^2=a^2$

$\Rightarrow$ y varies from 0 to $\sqrt{a^2-x^2}$ and x varies from 0 to a.

$$\therefore \iint xy\,dx\,dy = \int_0^a \int_0^{\sqrt{a^2-x^2}} xy\,dx\,dy$$

$$= \int_0^a \left[\int_0^{\sqrt{a^2-x^2}} xy\,dy \right] dx$$



$$= \int_0^a \left[x \frac{y^2}{2} \right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \frac{x(a^2-x^2)}{2}\,dx = \frac{a^4}{8}$$

Double integral polar co-ordinates :—

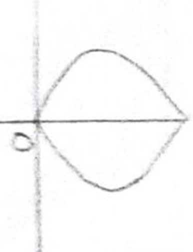
To evaluate the double integral in polar co-ordinates first integrate $f(r,\theta)r$ with respect to r keeping θ constants between the limits $r=f_1(\theta)$ and $r=f_2(\theta)$. and integrate the remaining expression with respect to θ between $\theta=\alpha$, $\theta=\beta$.

$$\therefore \iint_R f(r,\theta)\,r\,d\theta\,dr = \int_\alpha^\beta \left[\int_{r=f_1(\theta)}^{f_2(\theta)} r f(r,\theta)\,dr \right] d\theta$$

E Evaluate $\iint r\sin\theta\,d\theta\,dr$ over the quadriode cardiod $r=a(1-\cos\theta)$ above the initial line.

The cardiod is symmetric above the initial line and passes through the pole o.

Above the initial line the region of integration is $r=0, r=a(1-\cos\theta)$ and $\theta=0, \theta=\pi$



$$\iint r\sin\theta\,d\theta\,dr = \int_{r=0}^{a(1-\cos\theta)} \int_{\theta=0}^{\pi} r\sin\theta\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi} \left[\int_0^{a(1-\cos\theta)} (r\sin\theta)\,dr \right] d\theta$$

$$= \int_{\theta=0}^{\pi} \sin\theta \left(\frac{r^2}{2}\right)^{a(1-\cos\theta)} d\theta$$

$$= \frac{1}{2} \int_0^{\pi} \sin\theta \left[a^2(1-\cos\theta)^2\right] d\theta$$

$$= \frac{a^2}{2}\left[\frac{(1-\cos\theta)^3}{3}\right]_0^{\pi}$$

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$1-\cos\theta = u$

$\sin\theta\, d\theta = du$

$= \frac{a^2}{2}\int_0^{\pi} u^2\, du$

$= \frac{a^2}{2}\left[\frac{u^3}{3}\right]_0^{\pi}$

$\int_0^{\pi} u^2$

$$= \frac{a^2}{6}\left[(1-\cos\pi)^3 - (1-\cos 0)^3\right]$$

$1+1=(2)^3=8$

$$= \frac{a^2}{6}(8-0)$$

$$= \frac{4a^2}{3}$$

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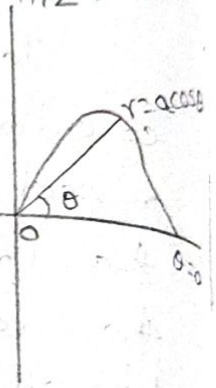
P Evaluate $\iint r\sqrt{a^2-r^2}\,d\theta\, dr$ over the upper half of the circle $r = a\cos\theta$



$$\iint r\sqrt{a^2-r^2}\,d\theta\,dr = \int_0^{\pi/2}\int_0^{a\cos\theta} r\sqrt{a^2-r^2}\,d\theta\,dr$$

$a^2-r^2 = t$

$-2r\,dr = dt$

$$= -\frac{1}{2}\int t^{1/2}\,dr \qquad r\,dr = -\frac{dt}{2}$$

$$= \frac{-1}{2}\,\frac{t^{3/2}}{3/2}$$

$$= \int_0^{\pi/2}\left[-\frac{1}{3}(a^2-r^2)^{3/2}\right]_0^{a\cos\theta} d\theta$$

$$= -\frac{1}{3}\int_0^{\pi/2}\left[(a^2-a^2\cos^2\theta)^{3/2} - (a^2)^{3/2}\right]d\theta$$

$$= \frac{a^3}{3}\;\; -\frac{1}{3}\int_0^{\pi/2}\left[(a^2)^{3/2}(1-\cos^2\theta)^{3/2} - (a^2)^{3/2}\right]d\theta$$

$$= \frac{a^3}{3}\int_0^{\pi/2}(1-\sin^3\theta)\,d\theta$$

$1-\cos^2\theta = \sin^2\theta$

$(1-\cos^2\theta)^{3/2}$

$=$

$$= \frac{a^3(3\pi-4)}{18}$$

Jacobian :—

In certain cases by changing the given variables x, y to new variables denoted by u, v given by the relations $x = f(u,v)$ $y = \phi(u,v)$. Evaluation of multiple integrals becomes easy.

Definition :—

If $u = f(x,y)$ and $v = \phi(x,y)$ be two continuous functions of the independent variables x and y such that $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are also continuous in x and y then

$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ is called the jacobian of u, v with respect to x, y and is denoted $J\left(\frac{u,v}{x,y}\right)$ or $\frac{\partial(u,v)}{\partial(x,y)}$

Definition :—

If u, v, w are functions of x, y, z the jacobian of u, v, w with respect to x, y, z is defined as

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

is denoted by

$J\left(\frac{u,v,w}{x,y,z}\right)$ or $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

Change of variables in double integrals :—

Transformation from cartesian to polar quadrant

Let the polar co-ordinates of point p whose cartesian co-ordinates are (x,y) be (r,θ) then

$x = r\cos\theta$, $y = r\sin\theta$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$\frac{d}{dx}(x^n) = nx^{n-1}$

$\frac{d(x)}{dx} = 1x^{1-1}$

(x, y)

$x = r\cos\theta$

$y = r\sin\theta$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$\frac{dx}{dr} = \frac{d(r\cos\theta)}{dr}$

$= \cos\theta$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r(1)$$

$$= r$$

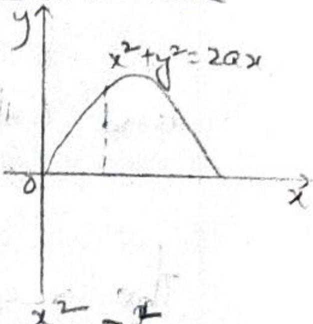
Hence $dx\,dy$ has to be changed $r\,dr\,d\theta$.

problems :-

By changing into polar co-ordinates evaluate the integral $\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} (x^2+y^2)\,dx\,dy$.

the region of integration is the semicircle $x^2+y^2 = 2ax$. $y=0$, $y=\sqrt{2ax-x^2}$

Above the x-axis $y^2 = 2ax - x^2$



changing into polar co-ordinates, $x^2+y^2=2ax$, the region becomes $r=0$, $r=2a\cos\theta$ $x^2, r^2 = 2ar\cos\theta$,

from $\theta = 0$ to $\theta = \pi/2$, hence the required integral $=$

$x = r\cos\theta$

$y = r\sin\theta$

$$= \int_0^{\pi/2}\int_0^{2a\cos\theta} \left[(r\cos\theta)^2 + (r\sin\theta)^2\right] r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^{2a\cos\theta} \left[r^2\cos^2\theta + r^2\sin^2\theta\right] r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^{2a\cos\theta} r^2\left[\cos^2\theta + \sin^2\theta\right] r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^{2a\cos\theta} r^3(1)\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^{2a\cos\theta} r^3\,dr\,d\theta$$

$$= \int_0^{\pi/2} \left[\frac{r^4}{4}\right]_0^{2a\cos\theta} d\theta$$

$$= \int_0^{\pi/2} \frac{\overset{4}{\cancel{16}}\, a^4 \cos^4\theta}{\cancel{4}}\, d\theta$$

$$= 4a^4 \int_0^{\pi/2} \cos^4\theta \, d\theta$$

$$= \frac{3\pi a^4}{4}$$

I changing into polar co-ordinates and evaluate

(4) $\int_0^a \int_0^{\sqrt{a^2-x^2}} e^{-(x^2+y^2)}\, dx\,dy.$

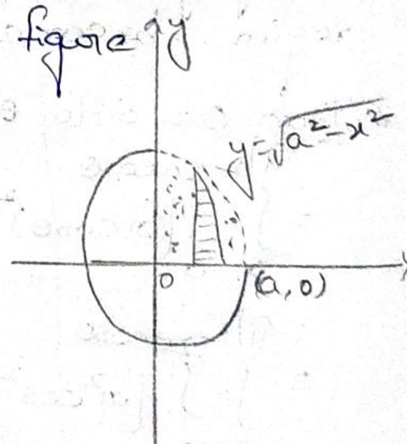
Here $\int_0^a \int_0^{\sqrt{a^2-x^2}} e^{-(x^2+y^2)}\, dx\,dy$

$f(x,y) = e^{-(x^2+y^2)}$

and the region bounded by $y=0, y=\sqrt{a^2-x^2}$

$x=0, x=a$

The region is shown in the figure



$x = r\cos\theta$

$y = r\sin\theta$

$f(x,y) = e^{-(r^2\cos^2\theta + r^2\sin^2\theta)}$

$= e^{-(r^2)(\cos^2\theta+\sin^2\theta)}$

$= e^{-r^2}(1)$

$= e^{-r^2}$

$y^2 = a^2 - x^2$

$x^2+y^2 = a^2$

the curve $y=\sqrt{a^2-x^2}$ is a circle whose equation is $x^2+y^2=a^2$.

the limits of $r = 0$ to a and the limits of $\theta = 0$ to $\pi/2$

thus $\int_0^a \int_0^{\sqrt{a^2-x^2}} e^{-(x^2+y^2)}\,dx\,dy = \int_0^{\pi/2} \int_0^a e^{-r^2} r\,dr\,d\theta$

put $r^2 = t$

$2r\,dr = dt$

$r\,dr = \frac{dt}{2}$ or $\frac{1}{2}dt$

$= \int_0^{\pi/2} \left[\int_0^a \frac{1}{2}(e^{-t})dt\right] d\theta$

$= \frac{1}{2}\int_0^{\pi/2} (e^{-t})_0^a \, d\theta$

$= \frac{1}{2}\int_0^{\pi/2} \left[-e^{-r^2}\right]_0^a d\theta$

$= \frac{1}{2}\int_0^{\pi/2} \left[-e^{-a^2}+1\right] d\theta$

$= \frac{1}{2}(-e^{-a^2}+1)\left[\theta\right]_0^{\pi/2}$

$= \frac{1}{2}(-e^{-a^2}+1)\left(\frac{\pi}{2}-0\right)$

$= \frac{\pi}{4}\left[1-e^{-a^2}\right]$.

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P using the transformation $x+y=u$, $x-y=v$, evaluate $\iint e^{\frac{x-y}{x+y}}\,dx\,dy$ over the region bounded by $x=0$, $y=0$ & $x+y=1$

Given $x+y=u$, $x-y=v$ solving these equations

$x+\cancel{y}=u$

$x-\cancel{y}=v$

$2x = u+v$

$x = \frac{u+v}{2}$

$x+y=u$

$x-y=v$

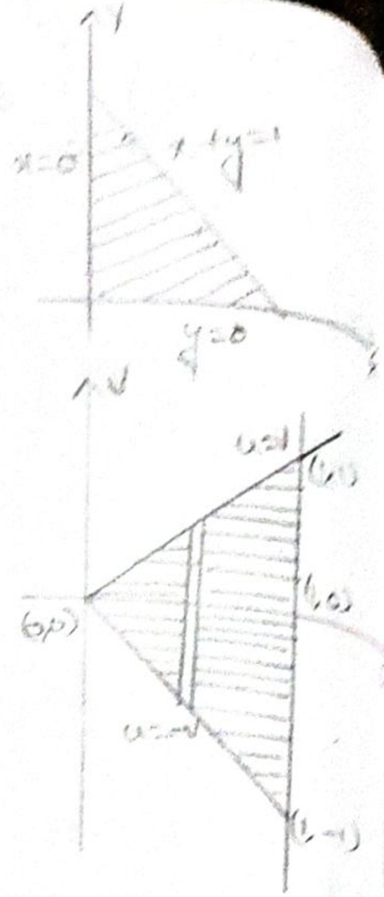
$2y = u-v$

$y = \frac{u-v}{2}$

$x = \frac{u+v}{2}$, $y = \frac{u-v}{2}$

We transform the function curves into uv plane with this substituation.

Curves	xy-plane	uv plane
	$x=0$	$u=-v$
	$y=0$	$u=v$
	$x+y=1$	$u=1$



$x=\frac{u+v}{2}, \; y=\frac{u-v}{2}$

$x+y=u, \; x-y=v$

$f(x,y)=e^{\frac{x-y}{x+y}}=e^{v/u}$

$$J\left(\frac{xy}{uv}\right)=\begin{vmatrix}\frac{dx}{du} & \frac{dx}{dv}\\ \frac{dy}{du} & \frac{dy}{dv}\end{vmatrix}$$

$$=\begin{vmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2}\end{vmatrix}$$

$$=-\frac{1}{4}-\frac{1}{4}$$

$$=\frac{-2}{\cancel{8}4}=\frac{-1}{2}$$

In the new region of u-v plane imagine a vertical strip with one end on $v=-u$ and other end on $v=u$. this has to be slided from $u=0$, to $u=1$, to cover the region. with the transformation the integral becomes

$dx\,dy=|J|\,du\,dv$

$=-\frac{1}{2}du\,dv$

$=\frac{1}{2}du\,dv$

$$=\int_0^1\int_{-u}^{u} e^{v/u}\,\frac{1}{2}\,du\,dv$$

$$=\frac{1}{2}\int_0^1\left[\int_{-u}^{u} e^{v/u}\,dv\right]du$$

$$=\frac{1}{2}\int_0^1\left(\frac{e^{v/u}}{1/u}\right)_{-u}^{u}du$$

$$= \frac{1}{2}\int_0^1 \left[u e^{v/u}\right]_{-u}^{u} du$$

$$= \frac{1}{2}\left[\int_0^1 \left(u e^{u/u} - u e^{-u/u}\right) du\right]$$

$$= \frac{1}{2}\int_0^1 \left[ue - ue^{-1}\right] du$$

$$= \frac{1}{2}\int_0^1 \left[ue - \frac{u}{e}\right] du$$

$$= \frac{1}{2}\left[u\ u\right]_0^1$$

$$= \frac{1}{2}\left[e\left[\frac{u^2}{2}\right]_0^1 - \frac{1}{e}\left(\frac{u^2}{2}\right)_0^1\right]$$

$$= \frac{1}{2}\left[\frac{e}{2} - \frac{1}{2e}\right]$$

$$= \frac{1}{4}\left[e - \frac{1}{e}\right]$$

P. By changing the variables evaluate

R 10M $\iint_R (x+y)^2\,dx\,dy$ where R is the region bounded

R by the parallelogram $x+y=0$, $x+y=1$, $2x-y=0$, $2x-y=4$.

Given $x+y=0$, $x+y=1$, $2x-y=0$, $2x-y=4$

put $x+y=u$, $2x-y=v$

Then the given parallelogram reduces a rectangle.

$$\begin{aligned} 2x - y &= v \\ x + y &= u \\ \hline 3x &= v+u \end{aligned}$$

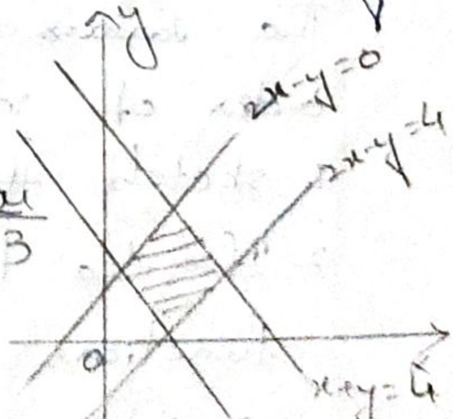
$$x = \frac{v+u}{3}$$

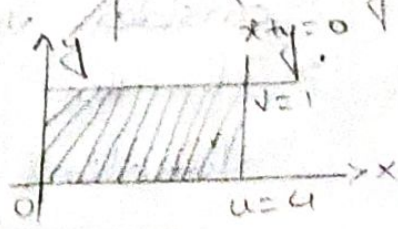
$$\begin{aligned} 2x - y &= v \\ 2x + 2y &= 2u \\ \hline x - 3y &= v-u \end{aligned}$$

$$y = \frac{v-2u}{-3}$$

$$-2y = \frac{v-u}{x}$$

$$y = \frac{v-u}{-2x}$$





$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = 1(-1) - 1(2) = -1 - 2$$

$$= -3$$

$$\therefore J = \frac{-1}{3}$$

$\therefore u$ varies from 0 to 1 and v varies from 0 to u

$$\therefore \iint_R (x+y)^2 \, dx\, dy$$

$$= \int_{u=0}^{1} \int_{v=0}^{u} u^2 \left(\frac{1}{3}\right) du\, dv$$

$$= \frac{1}{3} \int_{v=0}^{u} \left(\frac{u^3}{3}\right) dv$$

$$= \frac{1}{9} \int_{0}^{u} dv$$

$$= \frac{u}{9}$$

3/01/23

Change of order of Integration :—

Some times changing the order of integration makes the evaluation of integral easy. The various steps involve in changing the order of integration are.

1. sketch the region of integration.
2. If the limits on the inner integral are functions of y (with horizontal strip), Now consider vertical strip.
3. If the limits on the inner integral functions of x (with vertical strip)

now consider horizontal strip.

4. Find the new limits and evaluate.

change the order of integration and evaluate

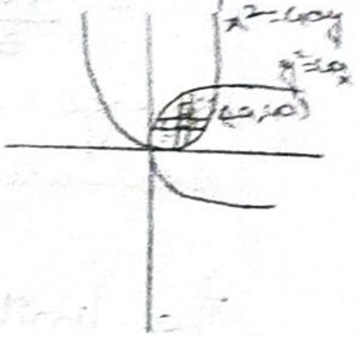
5. $\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dx\,dy.$

2

10M

the limits on the inner integral are functions of x and these are y limits, these limits are obtained by considering vertical strip

the region is bounded by



$y=\frac{x^2}{4a}, y=2\sqrt{ax}, x=0, x=4a$

and the region is shown in the fig

the point of intersection of the curve is $(4a,4a)$.

the change of order of integration, we consider horizontal strip with one end on $y^2=4ax$ and other end on $x^2=4ay \Rightarrow x=2\sqrt{ay}$ $x=\frac{y^2}{4a}$

Hence the $\iint$ with new limits is

$$\int_0^{4a}\int_{y^2/4a}^{2\sqrt{ay}} dx\,dy = \int_0^{4a}\left[\int_{y^2/4a}^{2\sqrt{ay}} dx\right]dy$$

$$= \int_0^{4a} [x]_{y^2/4a}^{2\sqrt{ay}}\,dx$$

$$= \int_0^{4a}\left(2\sqrt{ay}-\frac{y^2}{4a}\right)dx$$

$$=2\sqrt{a}\int_0^{4a} y^{1/2}dy - \frac{1}{4a}\int_0^{4a} y^2 dy$$

$$=2\sqrt{a}\left[\frac{y^{3/2}}{3/2}\right]_0^{4a} - \frac{1}{4a}\left[\frac{y^3}{3}\right]_0^{4a}$$

$= \frac{4}{3}\sqrt{a}\left[(4a^{3/2}) - 0\right] - \frac{1}{4}\left[\frac{64a^3}{3}\right]$

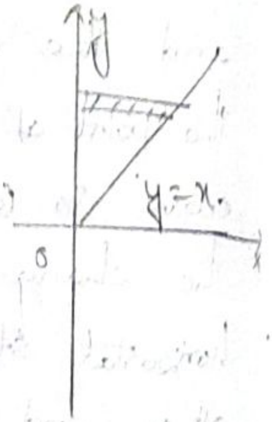
$= \frac{4}{3}\sqrt{a}\left[(2^2)^{3/2} a^{3/2}\right] - \frac{16a^2}{3}$

$= \frac{4}{3}\times a^2 \times 8 - \frac{16}{3}a^2$

$= \frac{32}{3}a^2 - \frac{16}{3}a^2$

$= \frac{16a^2}{3}$

I Change the order of integration and evaluate $\int_0^\infty\int_x^\infty \frac{e^{-y}}{y}\,dx\,dy$.



The limits of integration are given by the straight line $y=x$, $y=\infty$, $x=0$, $x=\infty$.

the region of integration is bounded by $x=0$, $y=x$ and an infinite boundary.

Hence taking the strips parallel to x-axis, the limits for y are from 0 to ∞ and hence order of integration We have

$\int_0^\infty \frac{e^{-y}}{y}\,dx\,dy = \int_0^\infty\int_0^y \frac{e^{-y}}{x} - 10$

$= \left[\int_0^y \left(\frac{e^{-x}}{y}\right) dx\right] dy$

$= \int_0^\infty \frac{1}{y}\left[e^{-x}\right]_0^y\, dy$

$= \int_0^\infty \frac{1}{y}(-e^{-y}-1)\,dy$

$$= \int_0^\infty \int_0^y \frac{e^{-y}}{y}\, dx\, dy$$

$$= \int_0^\infty \left[\int_0^y \frac{e^{-y}}{y}\, dx\right] dy$$

$$= \int_0^\infty \frac{e^{-y}}{y} \left[x\right]_0^y dy$$

$$= \int_0^\infty \frac{e^{-y}}{y} (y-0)\, dy$$

$$= \int_0^\infty e^{-y} dy$$

$$= \left(-e^{-y}\right)_0^\infty$$

$$= 0+1$$

Q Changing the order of integration and evaluate $\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy\, dy\, dx$

the limits on the inner integral.
the region is bounded by $y = \frac{x^2}{a}$, $y = 2a-x$.
to change the order of $x=0$, $x=a$.
integration. we consider
horizontal strip since for the
region